

A Mixed Boundary Value Problem That Arises in the Study of Adhesively Bonded Structures

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Given a physics-based problem, we often have choices in deciding

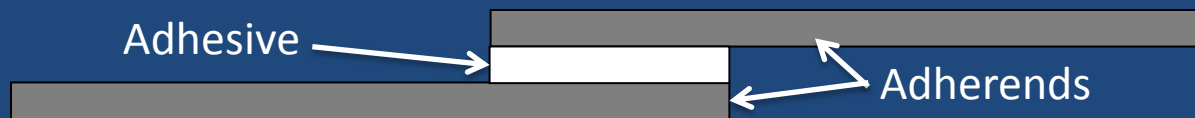
a) which Reduced-Order Model to use

b) which mathematical method to use to analyze the ROM

Examples come from Ocean Modeling, Electromagnetism, and Structural Mechanics

Typical Problem Genesis:

- Adhesively Bonded Joints



- Aerospace Sandwich Structures



Boundary Value Problem: Mixed form in terms of Stress, Infinitesimal Strain, and Displacement

The diagram shows a rectangular domain in the x - z plane. The vertical axis is labeled z and the horizontal axis is labeled x . The domain is bounded by $0 \leq x \leq L$ and $0 \leq z \leq \eta$. The boundary conditions are:

- Top boundary ($z = \eta$): $u(x, \eta) = u_t(x)$ and $w(x, \eta) = w_t(x)$
- Bottom boundary ($z = 0$): $u(x, 0) = u_b(x)$ and $w(x, 0) = w_b(x)$
- Left boundary ($x = 0$): $\sigma_{xx} = 0$ and $\sigma_{xz} = 0$
- Right boundary ($x = L$): $\sigma_{xx} = 0$ and $\sigma_{xz} = 0$

The governing equations within the domain are:

$$\nabla \cdot \underline{\underline{\sigma}} = \underline{\underline{0}} \quad \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}$$

Boundary Value Problem: Expressed in terms of Displacement

Diagram illustrating a Boundary Value Problem (BVP) expressed in terms of Displacement, showing a rectangular domain in the x - z plane. The domain is defined by $0 \leq x \leq L$ and $0 \leq z \leq \eta$.

The governing equations within the domain are:

$$(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial z^2} + (\lambda + G) \frac{\partial^2 w}{\partial x \partial z} = 0$$

$$(\lambda + 2G) \frac{\partial^2 w}{\partial z^2} + G \frac{\partial^2 w}{\partial x^2} + (\lambda + G) \frac{\partial^2 u}{\partial x \partial z} = 0$$

The boundary conditions are:

- Top boundary ($z = \eta$): $u(x, \eta) = u_t(x)$ and $w(x, \eta) = w_t(x)$
- Bottom boundary ($z = 0$): $u(x, 0) = u_b(x)$ and $w(x, 0) = w_b(x)$
- Left boundary ($x = 0$): $(\lambda + 2G) \frac{\partial u}{\partial x}(0, z) + \lambda \frac{\partial w}{\partial z}(0, z) = 0$ and $\frac{\partial w}{\partial x}(0, z) + \frac{\partial u}{\partial z}(0, z) = 0$
- Right boundary ($x = L$): $(\lambda + 2G) \frac{\partial u}{\partial x}(L, z) + \lambda \frac{\partial w}{\partial z}(L, z) = 0$ and $\frac{\partial w}{\partial x}(L, z) + \frac{\partial u}{\partial z}(L, z) = 0$

Boundary Value Problem: Expressed in terms of Displacement Potential

Diagram illustrating a Boundary Value Problem for a rectangular domain in the x - z plane. The domain is defined by $0 \leq x \leq L$ and $0 \leq z \leq \eta$. The governing equation is $\nabla^4 \Psi = 0$.

Boundary conditions are specified as follows:

- Top boundary ($z = \eta$):
 - Left: $\frac{\partial \Psi}{\partial x}(x, \eta) = u_t(x)$
 - Right: $\frac{\partial \Psi}{\partial z}(x, \eta) = w_t(x)$
- Bottom boundary ($z = 0$):
 - Left: $\frac{\partial \Psi}{\partial x}(x, 0) = u_b(x)$
 - Right: $\frac{\partial \Psi}{\partial z}(x, 0) = w_b(x)$
- Left boundary ($x = 0$):
 - Top: $(\lambda + 2G) \frac{\partial^2 \Psi}{\partial x^2}(0, z) + \lambda \frac{\partial^2 \Psi}{\partial z^2}(0, z) = 0$
 - Bottom: $\frac{\partial^2 \Psi}{\partial x \partial z}(0, z) = 0$
- Right boundary ($x = L$):
 - Top: $(\lambda + 2G) \frac{\partial^2 \Psi}{\partial x^2}(L, z) + \lambda \frac{\partial^2 \Psi}{\partial z^2}(L, z) = 0$
 - Bottom: $\frac{\partial^2 \Psi}{\partial x \partial z}(L, z) = 0$

$$\frac{\partial \Psi}{\partial x}(x, z) = u(x, z)$$

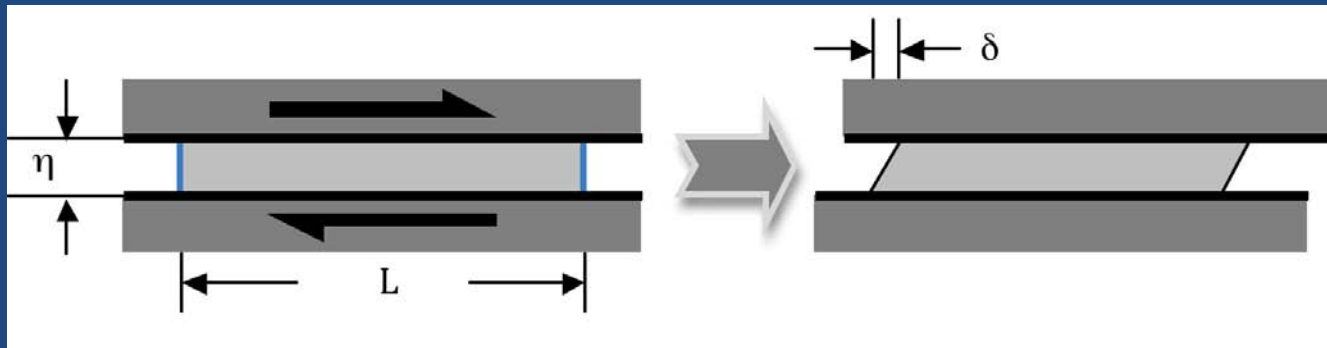
$$\frac{\partial \Psi}{\partial z}(x, z) = w(x, z)$$

Boundary Value Problem: Expressed in terms of Airy's Stress Function and Solution Ansatz

$$\begin{aligned}
 \nabla^4 \Phi &= 0; \{0 \leq x \leq L, 0 \leq z \leq \eta\} \\
 \frac{\partial^2 \Phi}{\partial z^2}(0, z) &= 0 \quad \frac{\partial^2 \Phi}{\partial x \partial z}(0, z) = 0 \\
 \frac{\partial^2 \Phi}{\partial z^2}(L, z) &= 0 \quad \frac{\partial^2 \Phi}{\partial x \partial z}(L, z) = 0 \\
 \left[\frac{1}{E} \int \left(\frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial z^2} \right) dz \right]_{z=0} + C_1(x) &= w_b(x) \\
 \left[\frac{1}{E} \int \left(\frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial z^2} \right) dz \right]_{z=\eta} + C_1(x) &= w_t(x) \\
 \left[-\frac{1}{G} \int \left(\frac{\partial^2 \Phi}{\partial x \partial z} \right) dz - \frac{1}{E} \iint \left(\frac{\partial^3 \Phi}{\partial x^3} - \nu \frac{\partial^3 \Phi}{\partial x \partial z^2} \right) dz dz \right]_{z=0} + C_2(x) &= u_b(x) \\
 \left[-\frac{1}{G} \int \left(\frac{\partial^2 \Phi}{\partial x \partial z} \right) dz - \frac{1}{E} \iint \left(\frac{\partial^3 \Phi}{\partial x^3} - \nu \frac{\partial^3 \Phi}{\partial x \partial z^2} \right) dz dz \right]_{z=\eta} + \eta \frac{dC_1}{dx}(x) + C_2(x) &= u_t(x)
 \end{aligned}$$

$$\Phi(x, z) = A_0(x) + zA_1(x) + \sum_{n=1}^{\infty} F_n(x) \sin\left(\frac{n\pi z}{\eta}\right)$$

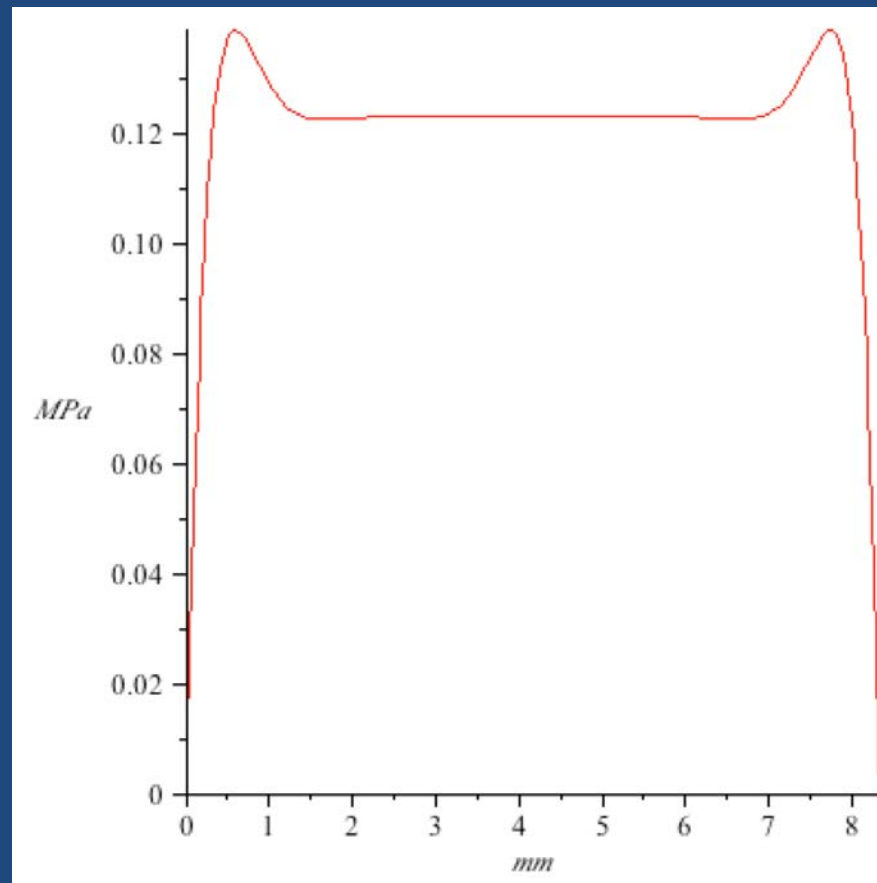
Example Loading Case: Displacements, Geometry, and Properties



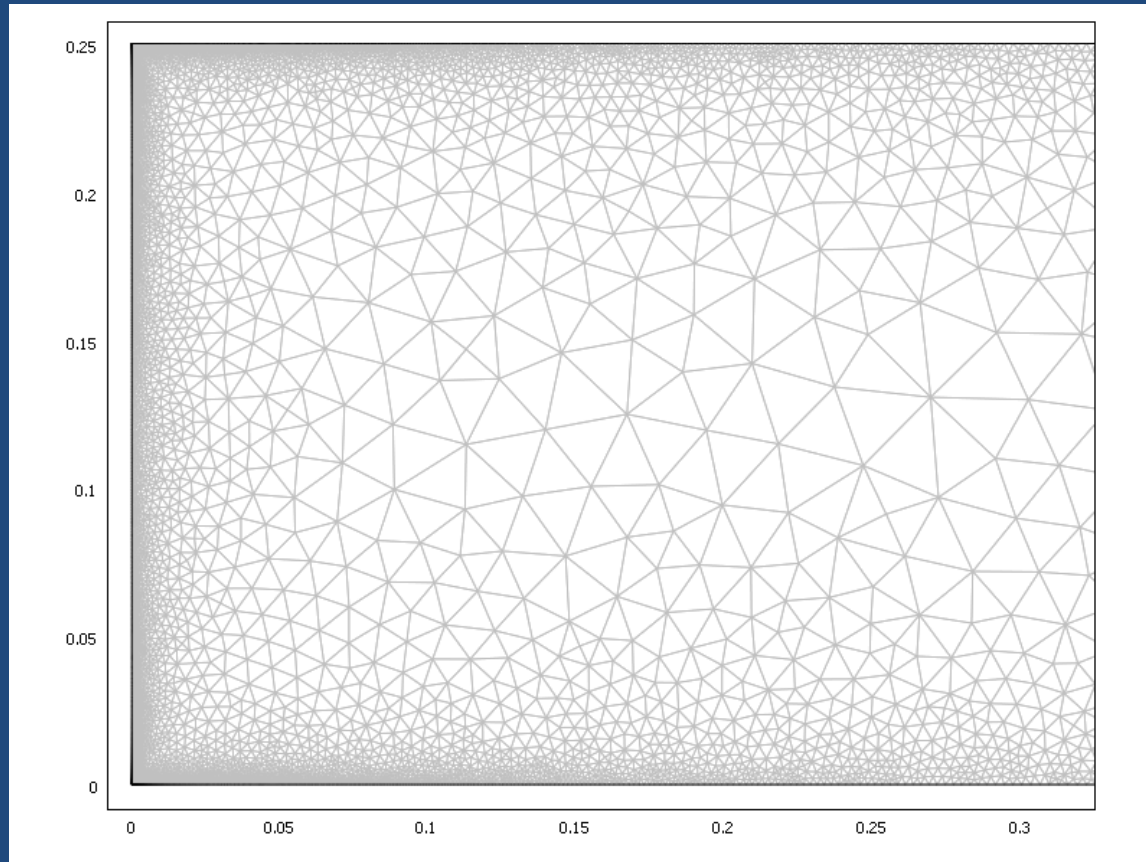
E	ν	L	η	δ
344.6 MPa	0.3	8.333 mm	0.25 mm	0.00025 mm

$u_b(x)$	$u_t(x)$	$w_b(x)$	$w_t(x)$
0 mm	0.00025 mm	0 mm	0 mm

Spectral-Collocation Analysis Results: Mid-plane Shear Stress $\sigma_{xz}(x, \eta/2)$ and Interfacial Shear Stresses $\sigma_{xz}(x, 0)$



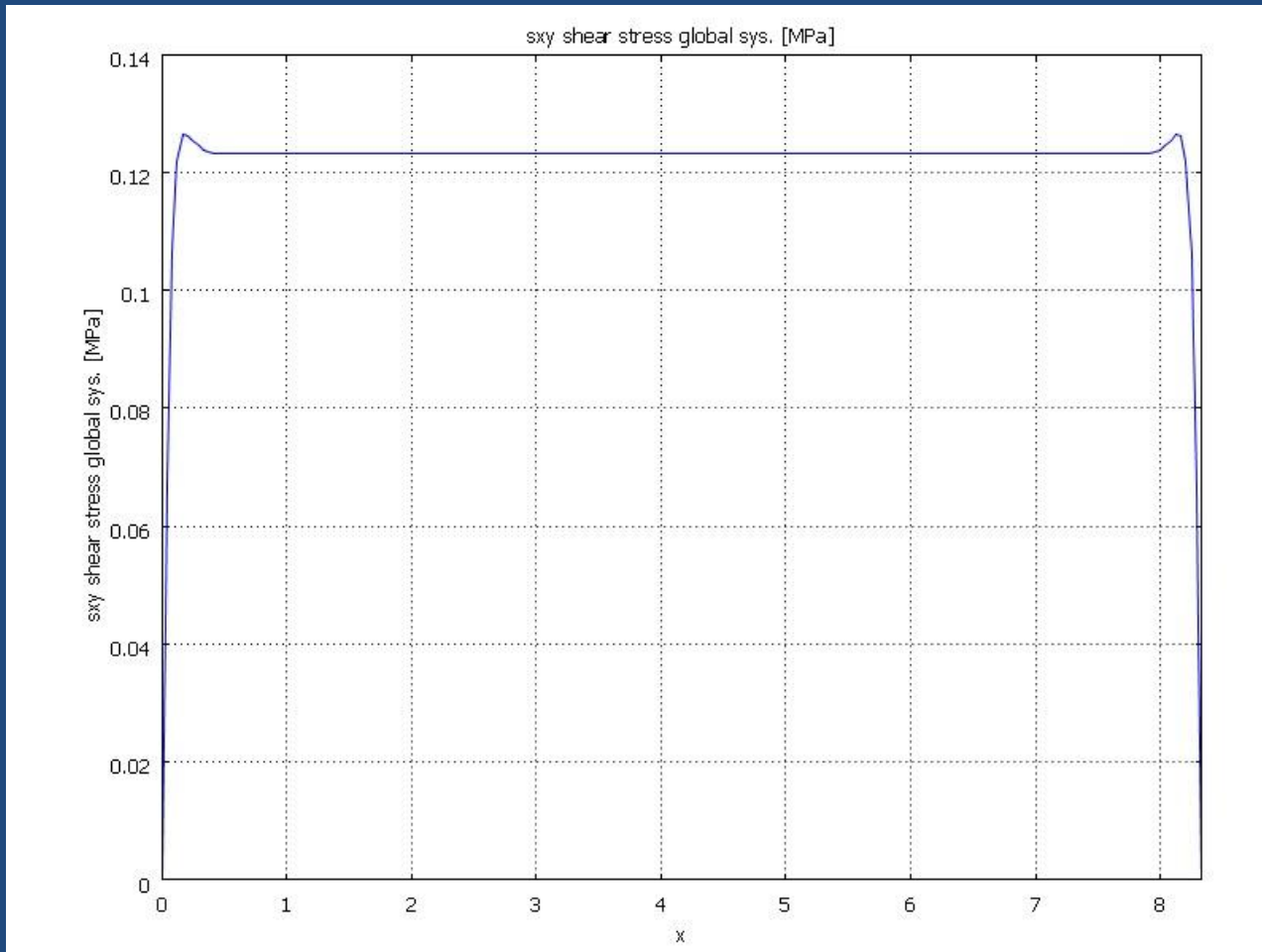
COMSOL Structural Mechanics Mesh: Shown in vicinity of stress free surface



COMSOL Structural Mechanics

2D Plane-Stress Analysis Results:

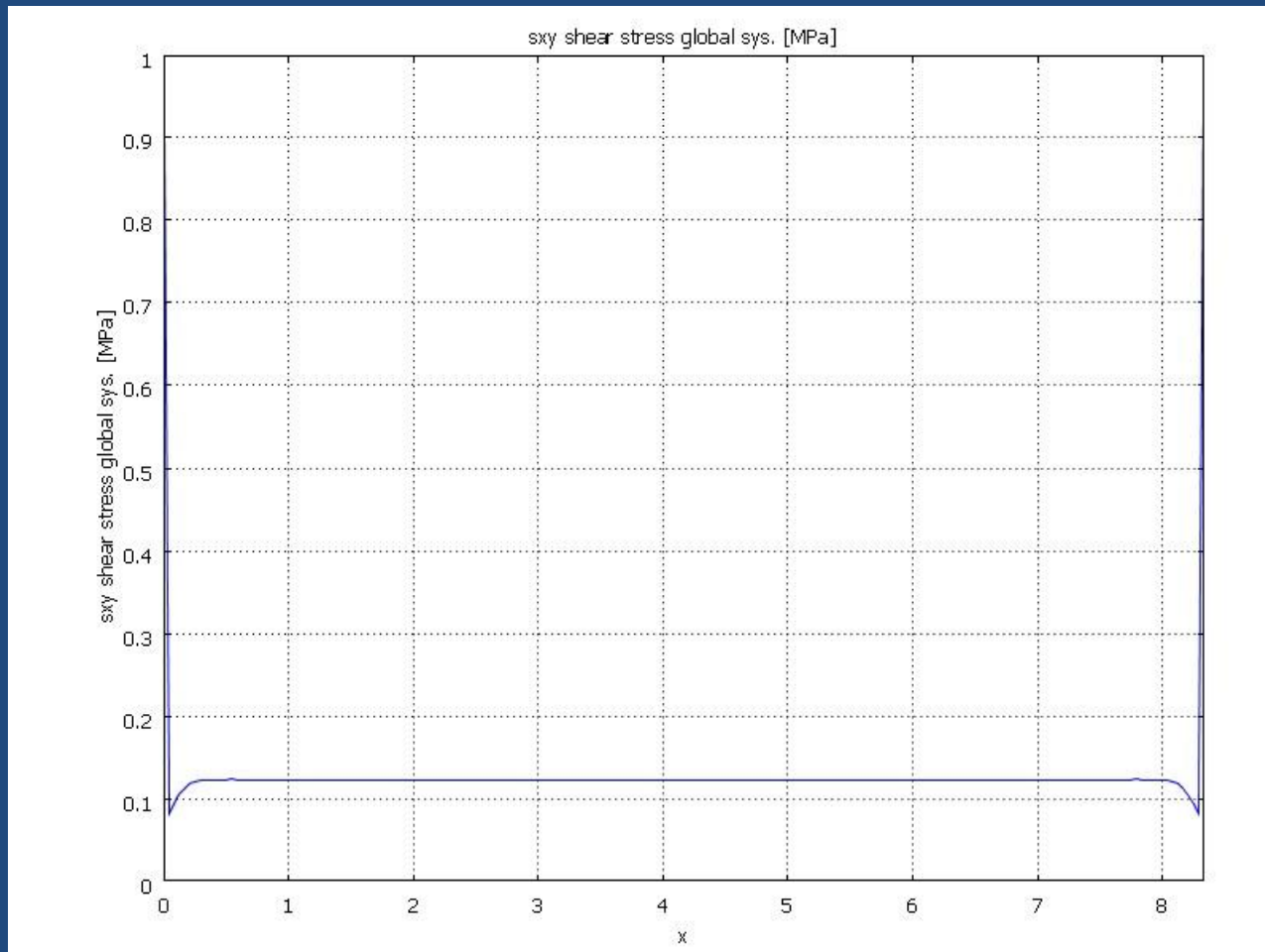
Mid-plane Shear Stress $\sigma_{xz}(x, \eta/2)$



COMSOL Structural Mechanics

2D Plane-Stress Analysis Results:

Interfacial Shear Stress $\sigma_{xz}(x,0)$



Back to Displacement BVP

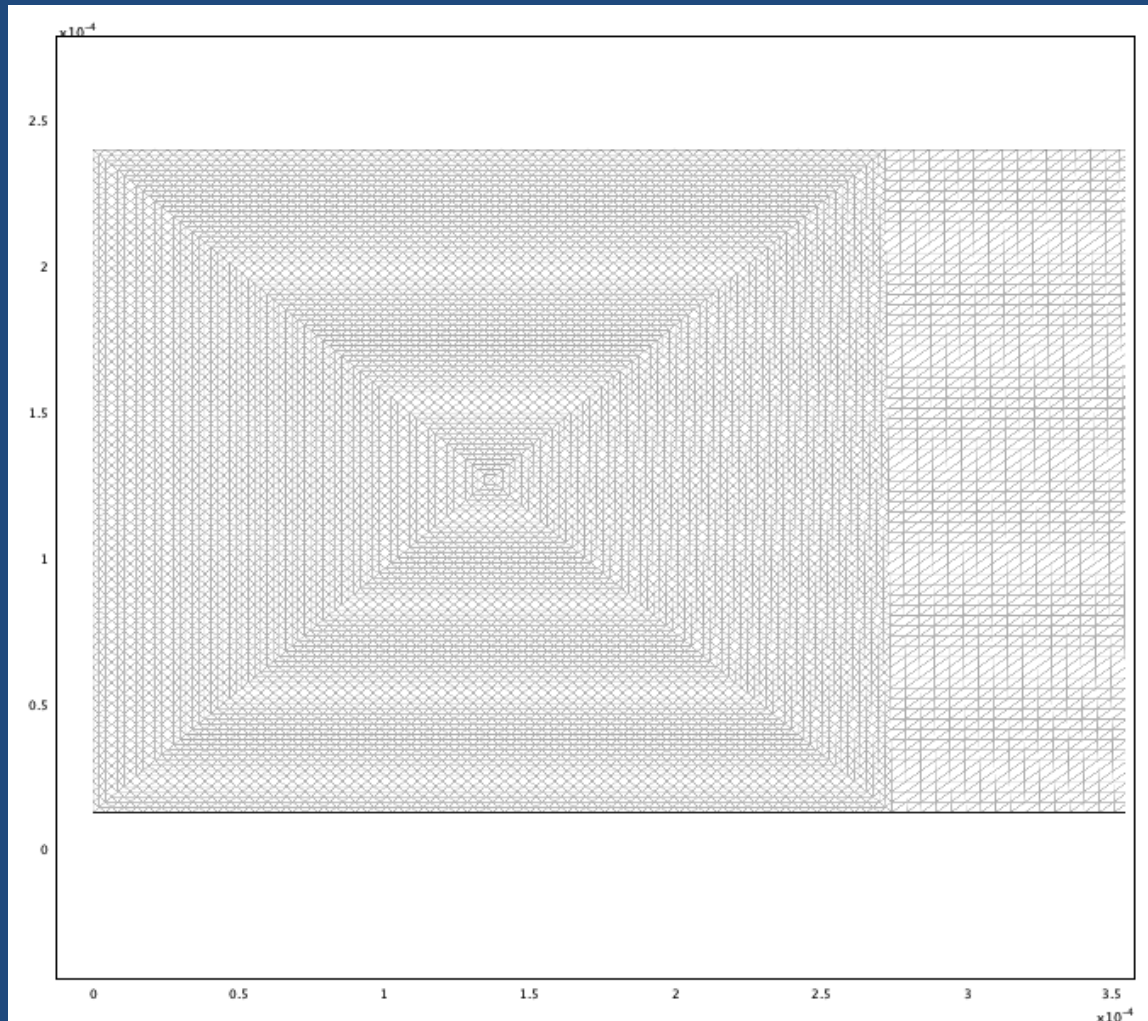
$u(x, \eta) = u_t(x) \quad w(x, \eta) = w_t(x)$
 $(\lambda + 2G) \frac{\partial u}{\partial x}(0, z) + \lambda \frac{\partial w}{\partial z}(0, z) = 0$
 $\frac{\partial w}{\partial x}(0, z) + \frac{\partial u}{\partial z}(0, z) = 0$
 $(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial z^2} + (\lambda + G) \frac{\partial^2 w}{\partial x \partial z} = 0$
 $(\lambda + 2G) \frac{\partial^2 w}{\partial z^2} + G \frac{\partial^2 w}{\partial x^2} + (\lambda + G) \frac{\partial^2 u}{\partial x \partial z} = 0$
 $(\lambda + 2G) \frac{\partial u}{\partial x}(L, z) + \lambda \frac{\partial w}{\partial z}(L, z) = 0$
 $\frac{\partial w}{\partial x}(L, z) + \frac{\partial u}{\partial z}(L, z) = 0$
 $u(x, 0) = u_b(x) \quad w(x, 0) = w_b(x)$

$$\nabla \cdot \boldsymbol{\Gamma} = \mathbf{F}$$

Use COMSOL's General PDE Solver

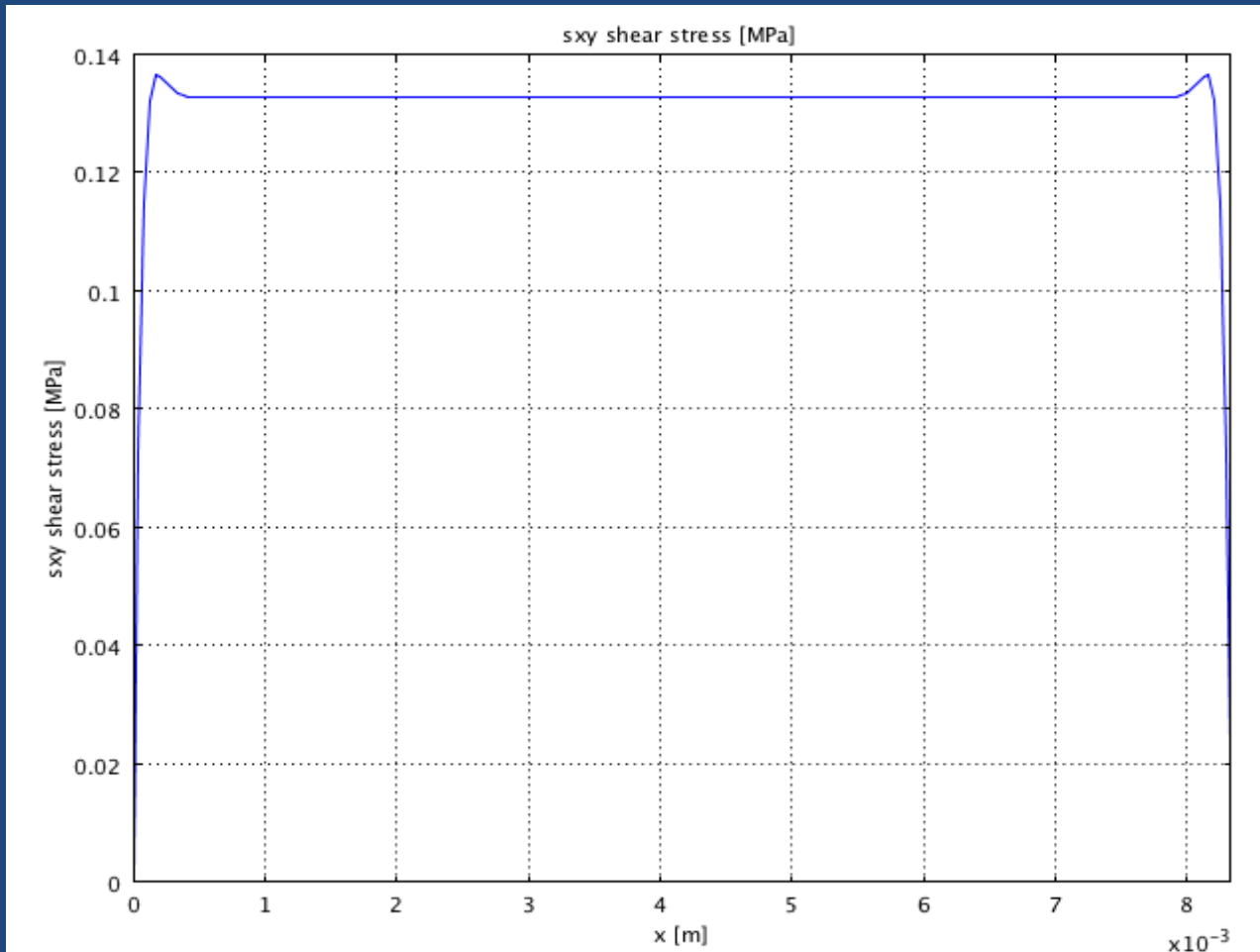
Gamma is a 2 x 2 tensor

COMSOL PDE (General Form) Solver Mesh: Shown in vicinity of stress free surface

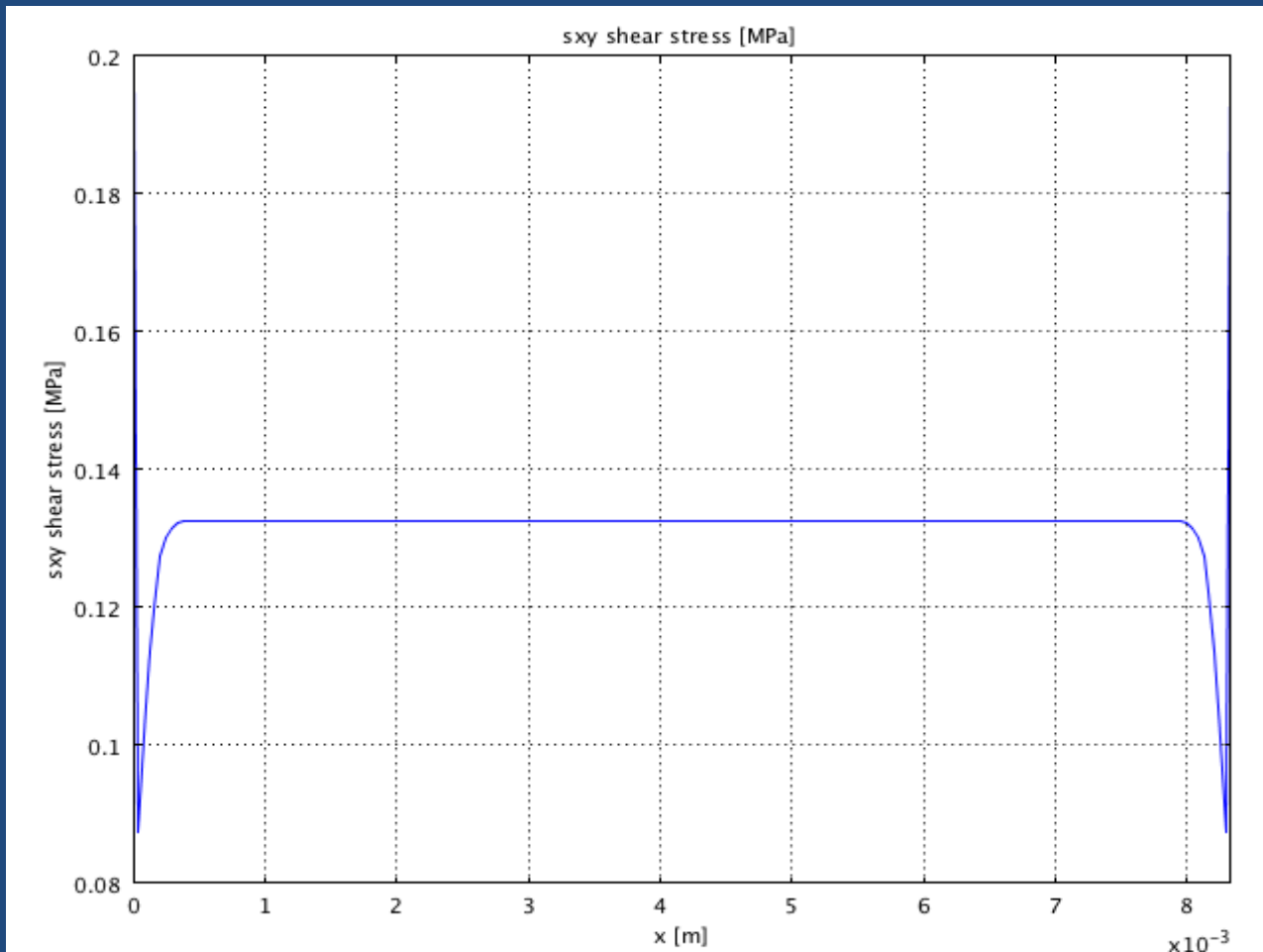


COMSOL PDE Solver Analysis Results:

Mid-plane Shear Stress $\sigma_{xz}(x, \eta/2)$



COMSOL PDE Solver Analysis Results: Interfacial Shear Stress $\sigma_{xz}(x,0)$



Some Conclusions/Observations

The Spectral Collocation method describes a shear stress that does not have a singularity at the corners; this seems to be the expected result from a mechanical point of view.

The Structural Mechanics result seems to suffer from artificially large singularities at the corners; did we implement it poorly?

The General PDE solver seems to be a natural way to pose this problem in COMSOL. It gives good result, although it seems to still suffer from some level of Numerical singularity at the corners.